18-819F: Introduction to Quantum Computing
47-779/785: Quantum Integer Programming & Quantum Machine Learning

First Look at Quantum Algorithms: Deutsch's Problem

Lecture 11

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Agenda

- What does computation mean in quantum circuits
  - Quantum parallelism – is it real?

- Deutsch’s problem
  - Deutsch’s circuit (algorithm)
  - Analysis of the Deutsch circuit

Quantum Computational Process

- Simply stated, in a computational process, we want a quantum computer to take a number x and produce another number $f(x)$ by way of some function f; we will think of f as applying a unitary transformation, U_f. Furthermore, U_f is reversible (in that it is its own inverse).

- We assume there is an input register with n qubits and an output register with m qubits in the computer.

- The action of the operator U_f on the computational basis states $|x\rangle_n|y\rangle_m$ of the input and output registers will be defined by

$$U_f(|x\rangle_n|y\rangle_m) = |x\rangle_n|y \oplus f(x)\rangle_m \quad \text{Eqn. (11.1).}$$

- The symbol $\oplus$ is addition modulo-2 and is equivalent to the exclusive OR operation we have already discussed. To illustrate its application, suppose in (11.1) that output register is $y = 0$, then (11.1) reduces to

$$U_f(|x\rangle_n|0\rangle_m) = |x\rangle_n|f(x)\rangle_m \quad \text{Eqn. (11.2).}$$

Quantum Computational Process

- To demonstrate the invertibility of U_f, we operate with it in (11.1) twice as follows

$$U_f U_f(|x\rangle|y\rangle) = U_f(|x\rangle|y \oplus f(x)\rangle) = |x\rangle|y \oplus f(x) \oplus f(x)\rangle = |x\rangle|y\rangle \quad \text{Eqn. (11.3).}$$

- Note that $f(x) \oplus f(x) = 0$ by definition of the exclusive OR operator we discussed in Lecture 10 (see the truth table for the exclusive OR operator in that Lecture).

- The Hadamard is one of the most important operators in quantum computing; it can be applied to 2-qubit and n-qubit states as follows

$$(H \otimes H)(|0\rangle \otimes |0\rangle) = (H|0\rangle)(H|0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \quad \text{Eqn. (11.4).}$$

- For an n-qubit state (11.4) generalizes to

$$H^{\otimes n}|0\rangle_n = \frac{1}{2^{n/2}} \sum_{0 \le x < 2^n} |x\rangle_n \quad \text{Eqn. (11.5).}$$

Quantum Parallelism

- From (11.4) and (11.5) we see that the Hadamard produces a superposition of the 2- or n-qubit input and output registers. If we then apply the unitary operator U_f, we see that the final state contains many evaluations of the function f at once. This is called quantum parallelism; it doesn't mean we have access to all the results of the evaluation. Measurement by the Born rules allows us to have only the values that collapse to the measurement basis.

- Application of U_f after H proceeds as follows

$$U_f\left(H^{\otimes n} \otimes 1_m\right)(|0\rangle_n|0\rangle_m) = \frac{1}{2^{n/2}} \sum_{0\le x<2^n} U_f(|x\rangle_n|0\rangle_m) \quad \text{Eqn. (11.6).}$$

- If we apply 20 Hadamard gates to the input before application of the operator U_f, then in theory the output will contain 2^{20} or over a million evaluations of the function f. These evaluation characterize the state of the output of the computation and only measurement will result the result and this causes a collapse into the measurement basis.

Deutsch's Problem

- This problem is about an unknown function f whose inputs are defined in $\{0,1\}$ and outputs in $\{0,1\}$. The question of interest is whether f is *balanced* or *constant*. Balanced means $f(0) \neq f(1)$ and constant means $f(0) = f(1)$.

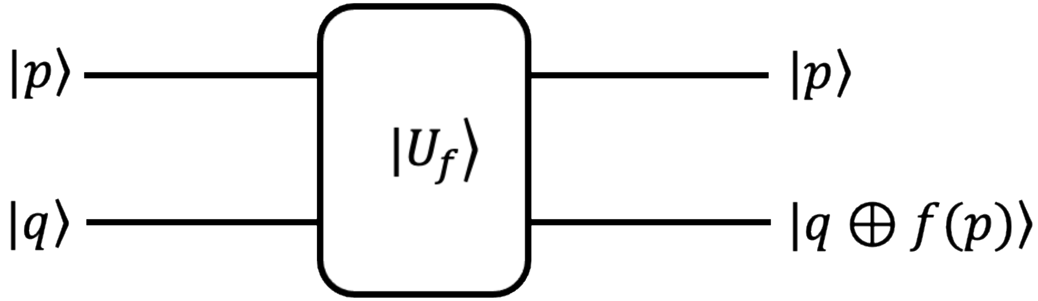
- The classical way to answer the question is to evaluate f for the inputs 0 and 1 and then check to see if $f(0) = f(1)$. At the minimum, one requires at least 2 evaluations: one for $f(0)$ and another for $f(1)$ to be able to give an answer. Deutsch wanted to know whether a quantum approach to the computation could answer the question more efficiently (fewer steps). In another words, with just fewer queries than the classical approach.

- This is an optimization problem, where the "cost function" is the number of queries to the operator U_f.

- Stating it this way if often known as the *quantum query complexity* model. This model one is given a box U_f and the interest is in how many times one has to query the box to get the answer one wants.

Definition of the Deutsch Problem

- We want an oracle U_f that can be questioned to determine if function f is constant or balanced. The truth table for the function is shown alongside the box representation of the oracle.



f	$f(0)$	$f(1)$
f_0	0	0
f_1	0	1
f_2	1	0
f_3	1	1

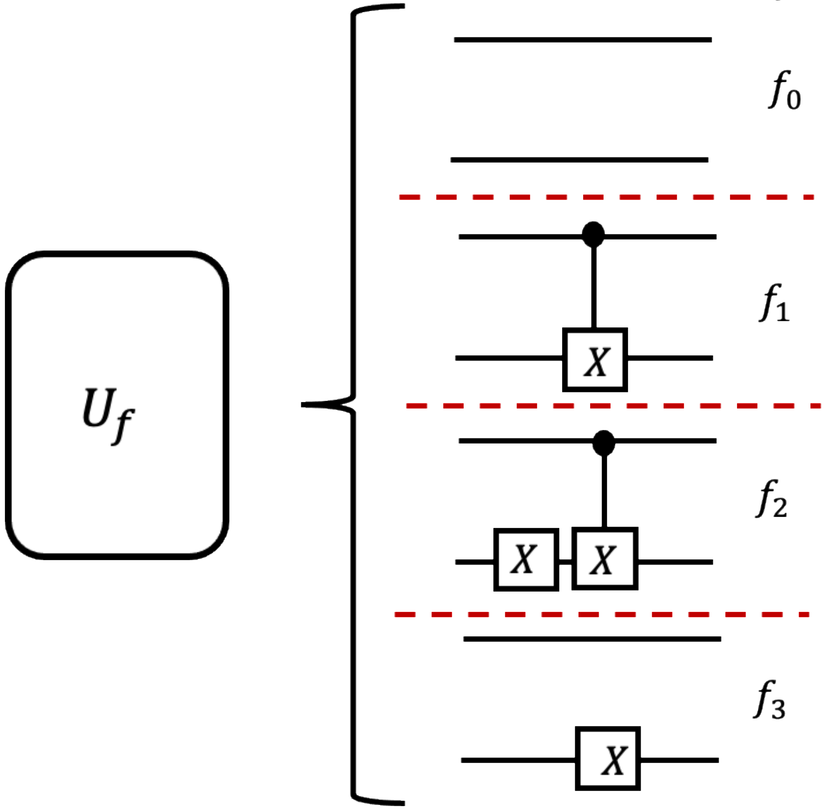
- The oracle performs the general computation given by the expression below.

$$U_f(|p\rangle|q\rangle) = |p\rangle|q \oplus f(p)\rangle$$

Possible Gate Circuit Implementations

- There are four distinct possible values that the function can be.
- If the input is $|0\rangle \otimes |0\rangle$ the output is $f_0 = |0\rangle|f(0)\rangle$;
- If the input is $|0\rangle \otimes |1\rangle$ the output is $f_1 = |0\rangle|1 \oplus f(0)\rangle$;
- If the input is $|1\rangle \otimes |0\rangle$ the output is $f_2 = |1\rangle|f(1)\rangle$;
- If the input is $|1\rangle \otimes |1\rangle$ the output is $f_3 = |1\rangle|1 \oplus f(1)\rangle$;
- Different gates as illustrate could achieve the output results. But we want a single circuit. As we will see Deutsch achieved this goal by using the Hadamard gate several times in his circuit.

Possible Gate Implementation of U_f



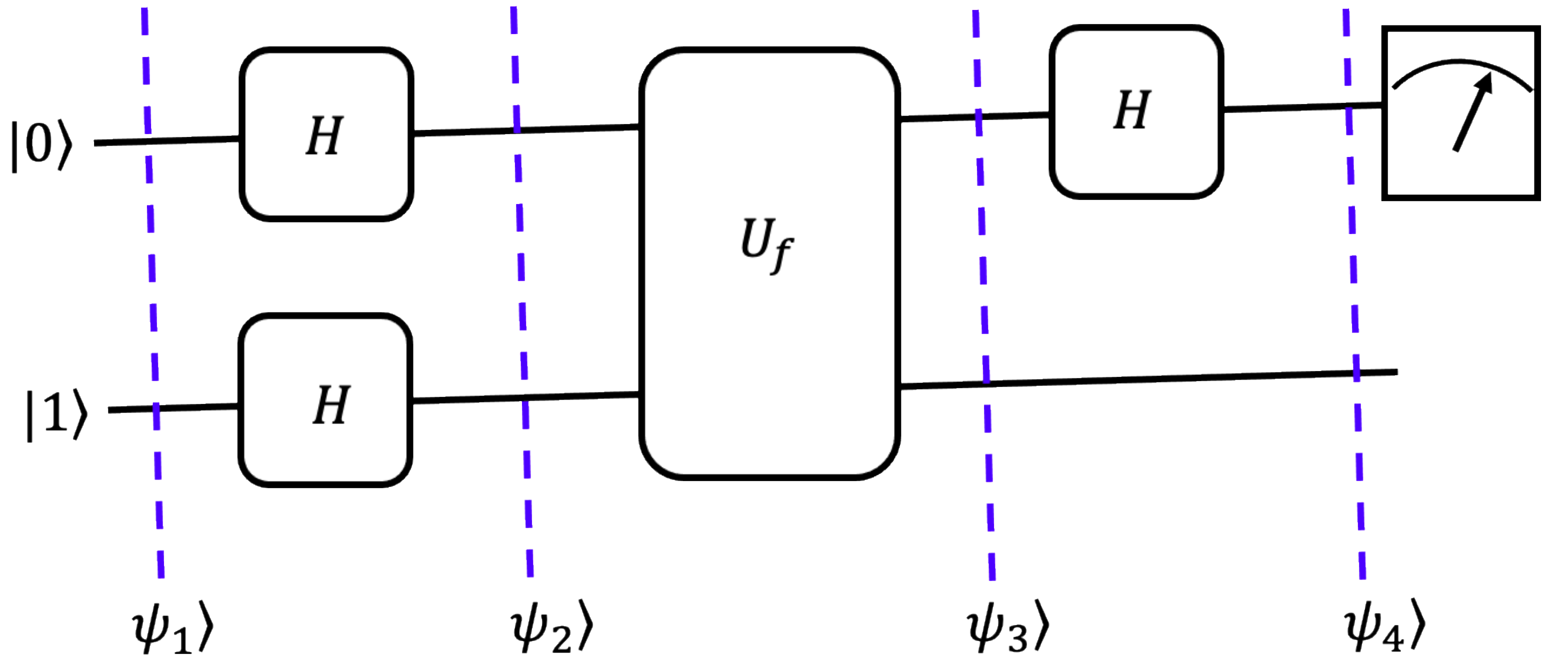
First Trial Solution to Deutsch's Problem

- A quantum mechanical way to approach the problem is to recall (11.1). However instead of f having just one qubit in the input register, we provide the input in superposition of the possible inputs {0,1}; We replace $|x\rangle$ in (11.1) with $\alpha|0\rangle + \beta|1\rangle$ and assume the output $|y\rangle$ is $|0\rangle$. When we operate with U_f, we expect the output to contain $f(0)$ and $f(1)$. Thus
  $|\psi\rangle_{out} = U_f(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle = \alpha U_f(|00\rangle) + \beta U_f(|10\rangle) = \alpha|0\rangle|f(0)\rangle + \beta|1\rangle\,|f(1)\rangle$ Eqn. (11.7).

- Note that we have used the fact that because $U_f|00\rangle = |0\rangle|0 \oplus f(0)\rangle$ we can write $0 \oplus f(0) = f(0)$, and for $U_f(|10\rangle) = |1\rangle|0 \oplus f(1)\rangle$ we can use $0 \oplus f(1) = f(1)$ to arrive at the last part of (11.7).

- Measurement in the $|0\rangle$ or $|1\rangle$ basis collapses the state into that one these basis and we still won't have an answer to the original problem of whether f is constant or balanced.

Circuit Deutsch's Algorithm



- The circuit that implements Deutsch's algorithm is shown above. We explain how it works in the following slides.

Analysis of the Deutsch Circuit

- The state at the input of the circuit is given by

$$|\psi_1\rangle = |01\rangle \quad \text{Eqn. (11.7).}$$

- After the Hadamard, the state of the system is given as

$$|\psi_2\rangle = |+\rangle|-\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \quad \text{Eqn. (11.8).}$$

- After applying the U_f operator the function is given by

$$|\psi_3\rangle = U_f|\psi_2\rangle = \frac{1}{2}\big(|0\rangle|f(0)\rangle - |0\rangle|1 \oplus f(0)\rangle + |1\rangle|f(1)\rangle + |1\rangle|1 \oplus f(1)\rangle\big) \quad \text{Eqn. (11.9).}$$

- If the function f is constant, then we have $f(0) = f(1)$, which allows us to simplify (11.9) to

- $$\left.\begin{aligned} |\psi_3\rangle &= \frac{1}{2}\big(|0\rangle|f(0)\rangle - |0\rangle|1 \oplus f(0)\rangle + |1\rangle|f(0)\rangle + |1\rangle|1 \oplus f(0)\rangle\big) \\ &= \frac{1}{2}[(|0\rangle + |1\rangle) \otimes |f(0)\rangle - (|0\rangle + |1\rangle) \otimes |1 \oplus f(0)\rangle] \\ &= \frac{1}{2}(|0\rangle + |1\rangle) \otimes |f(0)\rangle - |1 \oplus f(0)\rangle \\ &= \frac{1}{\sqrt{2}}|+\rangle \otimes (|f(0)\rangle - |1 \oplus f(0)\rangle) \end{aligned}\right\} \text{Eqn. (11.10).}$$

Analysis of the Deutsch Circuit

- From (11.10), we see that the first qubit has been transformed to the state $|+\rangle$.

- After the Hadamard, then the system state will be

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes (f(0)\rangle - |1 \oplus f(0)\rangle) \quad \text{Eqn. (11.11)}$$

- If we now measure the first qubit of (11.11), for sure the state will collapse to 0.
- If the function f is balanced, then $f(0) \neq f(1)$ and $f(0) \oplus 1 = f(1)$ and $f(1) \oplus 1 =$ f(0), then (11.9) can be simplified to

- $$\left.\begin{aligned} |\psi_3\rangle &= \tfrac{1}{2}\big(|0\rangle|f(0)\rangle - |0\rangle|f(1)\rangle + |1\rangle|f(1)\rangle - |1\rangle|f(0)\rangle\big) \\ &= \tfrac{1}{2}\big((|0\rangle - |1\rangle) \otimes |f(0)\rangle - (|0\rangle - |1\rangle) \otimes |f(1)\rangle\big) \\ &= \tfrac{1}{2}(|0\rangle - |1\rangle) \otimes (|f(0)\rangle - |f(1)\rangle) \\ &= \frac{1}{\sqrt{2}}|-\rangle \otimes \big(|f(0) - |f1)\rangle\rangle\big) \end{aligned}\right\} \text{Eqn. (11.12)}$$

Analysis of the Deutsch Circuit

- From (11.12), we see that the first qubit has been transformed to the state $|-\rangle$.
- After passing through the Hadamard, it is clear that

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}|1\rangle \otimes (|f(0)\rangle - |1 \oplus f(0)\rangle) \quad \text{Eqn. (11.11)} \quad \text{Eqn.(11.13)}$$

- We now see that the first qubit has been transformed to 1. When that is followed by measurement in the standard basis, we are assured that we will get 1.

- From (11.1) and (11.13) after measurement with a standard basis the circuit outputs, respectively, a 0 when f is constant and a 1 when f is balanced.

- The crucial point that Deutsch realized is that his algorithm can decide with just one query whether f is constant or balanced.

Utilizing a Phase Insight to Simplify Analysis

- The state vector $|+\rangle$ and $|-\rangle$ differ by a 180^o phase (as indicated by the minus sign).

- We investigate the action of the U_f on an input state that has $|-\rangle$ as a component, for example,

- $$\left.\begin{aligned} U_f|x\rangle|-\rangle &= \frac{1}{\sqrt{2}}\left(U_f|x\rangle|0\rangle - U_f|x\rangle|1\rangle\right) \\ &= \frac{1}{\sqrt{2}}(|x\rangle|f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle) \\ &= \frac{1}{\sqrt{2}}|x\rangle \otimes (|f(x)\rangle - |1 \oplus f(x)\rangle) \end{aligned}\right\} \text{Eqn. (11.14).}$$

- It is possible that $f(x) = 0$ or $f(x) = 1$; if $f(x) = 0$ then the last result in (11.14) becomes

$$U_f|x\rangle|-\rangle = |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |x\rangle|-\rangle \quad \text{Eqn. (11.15)}$$

- And if $f(x) = 1$ then the last result in (11.14) becomes

$$U_f|x\rangle|-\rangle = |x\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -|x\rangle|-\rangle \quad \text{Eqn. (11.16).}$$

- We can combine (11.15) and (11.16) and write $U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$ Eqn. (11.17).

Re-examining the Deutsch Circuit with Phase Insight

- The state vector of the circuit after the Hadamard can be re-written as

$$|\psi_2\rangle = |+\rangle|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|-\rangle)$$, Eqn. (11.18), where have not expanded $|-\rangle$.

- With the phase insight, the state vector, $|\psi_3\rangle$, after application of U_f becomes (using (11.17)

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left((-1)^{f(0)}|0\rangle|-\rangle + (-1)^{f(1)}|1\rangle|-\rangle\right)$$ Eqn. (11.19)

- For f constant $f(0) = f(1)$ and we can factor out $(-1)^{f(0)}$ and rewrite (11.19) as

$$|\psi_3\rangle = (-1)^{f(0)}\frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|-\rangle) = (-1)^{f(0)}|+\rangle|-\rangle$$ Eqn.. (11.20)

- When we apply the Hadamard the state vector $\psi_4\rangle$ becomes

$$|\psi_4\rangle = (-1)^{f(0)}|0\rangle|-\rangle$$ Eqn. (11.21).

- Measuring in the standard basis then yields 0 for the first qubit as before.
- Finally, when f is balanced and $f(0) \neq f(1)$ and we cannot factor out the -1; we must now write $|\psi_3\rangle$ as

$$|\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle|-\rangle - |1\rangle|-\rangle) = \pm|-\rangle|-\rangle$$ Eqn. (11.22).

Re-examining the Deutsch Circuit with Phase Insight

- Application of the last Hadamard to (11.22) gives us the expression for $|\psi_4\rangle$ as

$$|\psi_4\rangle = \pm|1\rangle|-\rangle \text{ Eqn. (11.23).}$$

- Measurement in the standard basis gives us the first qubit as 1 with an uncertainty. This is the same result we obtained earlier.

- Deutsch's problem was posed to determine if quantum computers could do some things more efficiently than classical computers.

- The question then was to find what is meant more efficiently. Time of execution is relevant parameter but not the only one.

- What Deutsch set out to prove was the *query complexity* aspect of computing. In this case the quantum process performed better and in some this advantage could translate into a ”faster” execution time.

Summary

- Quantum Computation
  - What does quantum parallelism mean
- Discussed Deutsch's problem and his circuit
  - Introduced the phase insight for simplifying quantum computational equations
- Discussed some helpful things about the Hadamard operator